QUESTION 1

Mihir has stayed up until 4 A.M. trying to solve these Ted-Ed riddles. Help him solve the riddles so he can finally go to sleep before he gets caught.

Sneaky Shreyas walks into China First and pays for \$30 dollars worth of food with a \$100 bill. However the cashier, Justin, doesn't have change for a \$100 dollar bill so he goes to Prabhas's Fish store to break the bill into 10 \$10 bills. Justin then gives adequate change to Shreyas, and Shreyas leaves the store. Ten minutes later, Prabhas walks in mad because the \$100 bill was counterfeit. Prabhas takes \$100 from Justin. Not including the value of the food, let A be the dollar amount of money Justin lost.

Dylan finds 9 identical pencils in his hair, but he can't figure out which one is his. He knows that his pencil weighs slightly more than the other pencils, so he gets out his handy balancing scale. Let B be the minimum number of weight comparisons that Dylan needs to make to guarantee that he finds his pencil.

Find $A + B$.

QUESTION 1

Mihir has stayed up until 4 A.M. trying to solve these Ted-Ed riddles. Help him solve the riddles so he can finally go to sleep before he gets caught.

Sneaky Shreyas walks into China First and pays for \$30 dollars worth of food with a \$100 bill. However the cashier, Justin, doesn't have change for a \$100 dollar bill so he goes to Prabhas's Fish store to break the bill into 10 \$10 bills. Justin then gives adequate change to Shreyas, and Shreyas leaves the store. Ten minutes later, Prabhas walks in mad because the \$100 bill was counterfeit. Prabhas takes \$100 from Justin. Not including the value of the food, let A be the dollar amount of money Justin lost.

Dylan finds 9 identical pencils in his hair, but he can't figure out which one is his. He knows that his pencil weighs slightly more than the other pencils, so he gets out his handy balancing scale. Let B be the minimum number of weight comparisons that Dylan needs to make to guarantee that he finds his pencil.

Find $A + B$.

QUESTION 2

Justin chooses a random real number x from the interval $[0, 2\pi]$. What is the probability that exactly two of the following statements are true? (Assume that if one or both sides of the inequality are undefined, then the statement is false for that value of x.)

$$\begin{aligned} A &= \cos(2x) + 2\sin^2(x) > \sin(x) + 1 \\ B &= \sec(x) > \csc(x) \\ C &= \sin(2x) + 1 > \sin(x) + 2\cos(x) \end{aligned}$$

QUESTION 2

Justin chooses a random real number x from the interval $[0, 2\pi]$. What is the probability that exactly two of the following statements are true? (Assume that if one or both sides of the inequality are undefined, then the statement is false for that value of x.)

$$\begin{aligned} A &= \cos(2x) + 2\sin^2(x) > \sin(x) + 1 \\ B &= \sec(x) > \csc(x) \\ C &= \sin(2x) + 1 > \sin(x) + 2\cos(x) \end{aligned}$$

QUESTION 3

$$
\begin{aligned}
X &= \text{The eccentricity of the equation defined by the parameters } x = t^2 + t + 1 \text{ and } y = t^2 - t + 1 \\
Y &= \text{The eccentricity of the equation defined by the parameters } x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) \text{ and } y = \frac{2t}{1+t^2}
\end{aligned}
$$

Find $X + Y$.

QUESTION 3

$$\begin{aligned} X &= \text{The eccentricity of the equation defined by the parameters } x = t^2 + t + 1 \text{ and } y = t^2 - t + 1 \\ Y &= \text{The eccentricity of the equation defined by the parameters } x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) \text{ and } y = \frac{2t}{1+t^2} \end{aligned}$$

Find $X + Y$.

QUESTION 4

A = Find the geometric mean of the arithmetic mean and the harmonic mean of 529 and 729

B = The minimum value of the function $f(x) = x^2 + \dfrac{1}{x}$ for positive values of x (Hint: Split the 1/x term)

Find $A \log_4 \left(\dfrac{B}{3}\right)$

QUESTION 4

A = Find the geometric mean of the arithmetic mean and the harmonic mean of 529 and 729

B = The minimum value of the function $f(x) = x^2 + \frac{1}{x}$ for positive values of x (Hint: Split the 1/x term)

Find $A \log_4 \left(\frac{B}{3}\right)$

QUESTION 5

Rohan's vector $\mathbf{r}$ has magnitude $4\sqrt{2}$ and Vishnav's vector $\mathbf{v}$ has magnitude $7\sqrt{2}$. The dot product of the two vectors is equal to the magnitude of their cross-product. Both vectors start at the origin and the angle between the vectors is acute.

A = The distance between the endpoints of the two vectors

B = The area of the triangle bounded by the two vectors and the line connecting their endpoints

C = The length of the projection of the smaller vector onto the larger one

Calculate $A^2 + 4B + C$

QUESTION 5

Rohan's vector $\mathbf{r}$ has magnitude $4\sqrt{2}$ and Vishnav's vector $\mathbf{v}$ has magnitude $7\sqrt{2}$. The dot product of the two vectors is equal to the magnitude of their cross-product. Both vectors start at the origin and the angle between the vectors is acute.

A = The distance between the endpoints of the two vectors

B = The area of the triangle bounded by the two vectors and the line connecting their endpoints

C = The length of the projection of the smaller vector onto the larger one

Calculate $A^2 + 4B + C$

QUESTION 6

A = The probability that $7^m + 7^n$ is divisible by 5, given that m and n are positive integers between 1 and 100 inclusive.

B = $\tan(3285^\circ)$

C = The area of the intersection between the graphs $r = \cos\theta$ and $r = \sin\theta$

Find $8B(A + C)$

QUESTION 6

A = The probability that $7^m + 7^n$ is divisible by 5, given that m and n are positive integers between 1 and 100 inclusive.

B = $\tan(3285^\circ)$

C = The area of the intersection between the graphs $r = \cos\theta$ and $r = \sin\theta$

Find $8B(A + C)$

QUESTION 7

At 3 pm on Monday, Dylan has 1 pencil behind his ear. Each day, the number of pencils behind his ear, x, changes at 3 pm. On the nth day after this Monday at 3 pm, Dylan randomly chooses to either add n pencils to his ear, add $9x + n$ pencils to his ear, or add $10^{\lfloor \log(x)+1 \rfloor} \cdot n$ pencils to his ear. After 3pm on Day 6, what is the probability that Dylan has over one hundred thousand pencils behind his ear?

QUESTION 7

At 3 pm on Monday, Dylan has 1 pencil behind his ear. Each day, the number of pencils behind his ear, x, changes at 3 pm. On the nth day after this Monday at 3 pm, Dylan randomly chooses to either add n pencils to his ear, add $9x + n$ pencils to his ear, or add $10^{\lfloor \log(x)+1 \rfloor} \cdot n$ pencils to his ear. After 3pm on Day 6, what is the probability that Dylan has over one hundred thousand pencils behind his ear?

QUESTION 8

Rohan rejected Farzan's 20 part question last year, so to get revenge he creates a problem with an infinite number of parts!

$$\begin{aligned}
P_1 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \frac{1}{p+q} \\
P_2 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^2 \\
P_3 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^3 \\
&\vdots \\
P_n &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^n
\end{aligned}$$

Find $\displaystyle\lim_{n\to\infty} \sum_{a=1}^{n} P_a$

QUESTION 8

Rohan rejected Farzan's 20 part question last year, so to get revenge he creates a problem with an infinite number of parts!

$$
\begin{aligned}
P_1 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \frac{1}{p+q} \\
P_2 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^2 \\
P_3 &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^3 \\
&\vdots \\
P_n &= \sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \left(\frac{1}{p+q}\right)^n
\end{aligned}
$$

Find $\lim_{n\to\infty} \sum_{a=1}^{n} P_a$

QUESTION 9

Let:

$$
\begin{aligned}
A &= \lim_{x\to 0}\left(1+\frac{4x}{(3x+1)^2}\right)^{\frac{2}{x}} \\
B &= \lim_{n\to\infty}\sum_{x=0}^{\infty}\frac{(2n)^x\cdot(\ln(n+1)-\ln(n))^x}{x!}
\end{aligned}
$$

Find $\ln(AB)$.

QUESTION 9

Let:

$$
\begin{aligned}
A &= \lim_{x\to 0}\left(1+\frac{4x}{(3x+1)^2}\right)^{\frac{2}{x}} \\
B &= \lim_{n\to\infty}\sum_{x=0}^{\infty}\frac{(2n)^x\cdot(\ln(n+1)-\ln(n))^x}{x!}
\end{aligned}
$$

Find $\ln(AB)$.

QUESTION 10

Dylan gets bored of precalculus halfway through the year, so he decides to learn calculus. Help him solve the following differentiation questions.

The function $f'(x)$ is defined as $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\begin{aligned} A &= f'(x) \text{ where } f(x) = x^2 \\ B &= f'(x) \text{ where } f(x) = \sqrt{x} \end{aligned}$$

The first 3 terms of a Maclaurin Series for a function $f(x)$ is defined as $\frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$ where $f''(x) = \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$

$$C = \text{The first 3 terms of the Maclaurin series for } f(x) = x^2 - 3x + 4$$

If $C + (AB)^2$ can be expressed as $px^2 - qx + r$, find $p + q + r$.

QUESTION 10

Dylan gets bored of precalculus halfway through the year, so he decides to learn calculus. Help him solve the following differentiation questions.

The function $f'(x)$ is defined as $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$\begin{aligned} A &= f'(x) \text{ where } f(x) = x^2 \\ B &= f'(x) \text{ where } f(x) = \sqrt{x} \end{aligned}$$

The first 3 terms of a Maclaurin Series for a function $f(x)$ is defined as $\frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$ where $f''(x) = \lim_{h\to 0} \dfrac{f'(x+h)-f'(x)}{h}$

$$C = \text{The first 3 terms of the Maclaurin series for } f(x) = x^2 - 3x + 4$$

If $C + (AB)^2$ can be expressed as $px^2 - qx + r$, find $p + q + r$.

QUESTION 11

Dylan still needs more help with calculus. Now help him solve the following integration questions.

The value of the integral $\int_n^k f(x)dx$ is defined as (Area bounded by $f(x)$, the lines $x = n$ and $x = k$ that is above the x-axis) $-$ (Area bounded by $f(x)$, the lines $x = n$ and $x = k$ that is below the x-axis)

$$\begin{aligned} A &= \int_{-1}^{2} (-|x|+1)dx \\ B &= \int_{0}^{1.5} \sqrt{9-4x^2}dx \\ C &= \int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} (\sin x + \cos x)^2 dx \end{aligned}$$

Compute $\dfrac{B}{AC}$.

QUESTION 11

Dylan still needs more help with calculus. Now help him solve the following integration questions.

> The value of the integral $\int_n^k f(x)dx$ is defined as
> (Area bounded by $f(x)$, the lines $x = n$ and $x = k$ that is above the x-axis) $-$
> (Area bounded by $f(x)$, the lines $x = n$ and $x = k$ that is below the x-axis)

$$\begin{aligned} A &= \int_{-1}^{2} (-|x| + 1)dx \\ B &= \int_{0}^{1.5} \sqrt{9 - 4x^2}dx \\ C &= \int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} (\sin x + \cos x)^2 dx \end{aligned}$$

Compute $\dfrac{B}{AC}$.

QUESTION 12

Evaluate:

$$\sqrt{\left|\begin{bmatrix} 8 & 3 & 24 & 10 \\ 4 & 5 & 12 & 5 \\ 2 & 4 & 8 & 9 \\ 1 & 12 & 3 & 2 \end{bmatrix}\begin{bmatrix} 16 & 8 & 4 & 2 \\ 6 & 10 & 8 & 24 \\ 48 & 24 & 16 & 6 \\ 20 & 10 & 18 & 4 \end{bmatrix}\right|}$$

QUESTION 12

Evaluate:

$$\sqrt{\left|\begin{bmatrix} 8 & 3 & 24 & 10 \\ 4 & 5 & 12 & 5 \\ 2 & 4 & 8 & 9 \\ 1 & 12 & 3 & 2 \end{bmatrix}\begin{bmatrix} 16 & 8 & 4 & 2 \\ 6 & 10 & 8 & 24 \\ 48 & 24 & 16 & 6 \\ 20 & 10 & 18 & 4 \end{bmatrix}\right|}$$

QUESTION 13

Farzan is studying electrical engineering and finds that the equivalent resistance for a set of n resistors $R_0, R_1, \cdots R_n$ is the reciprocal of the sum of their reciprocals. Farzan, however, has an infinite number of resistors! (Equivalent resistance is calculated in the same way, except the sum of the reciprocals is now an infinite series.)

$$\begin{aligned} A &= \text{The equivalent resistance of Farzan's resistors if } R_n \text{ is the } (n+1)^{th} \text{ triangular number for all } n \geq 0 \\ B &= \text{The equivalent resistance of Farzan's resistors if } R_n = \frac{(3^{n+1})}{n+1} \text{ for all } n \geq 0 \\ C &= \text{The equivalent resistance of Farzan's resistors if } R_0 = 1 \text{ and } R_n = \frac{n(R_{n-1})}{3} \text{ for all } n > 0 \end{aligned}$$

Compute ABC.

QUESTION 13

Farzan is studying electrical engineering and finds that the equivalent resistance for a set of n resistors $R_0, R_1, \cdots R_n$ is the reciprocal of the sum of their reciprocals. Farzan, however, has an infinite number of resistors! (Equivalent resistance is calculated in the same way, except the sum of the reciprocals is now an infinite series.)

A = The equivalent resistance of Farzan's resistors if R_n is the $(n+1)^{th}$ triangular number for all $n \geq 0$

B = The equivalent resistance of Farzan's resistors if $R_n = \dfrac{(3^{n+1})}{n+1}$ for all $n \geq 0$

C = The equivalent resistance of Farzan's resistors if $R_0 = 1$ and $R_n = \dfrac{n(R_{n-1})}{3}$ for all $n > 0$

Compute ABC.

QUESTION 14

$$\begin{aligned} F &= \text{The cube root of unity that lies in the 2nd Quadrant of the Argand plane.} \\ A &= \text{The 5th root of unity that lies in the 4th quadrant of the Argand plane} \\ M &= \operatorname{cis}(48^\circ) \end{aligned}$$

Find $\dfrac{FA}{M}$.

QUESTION 14

$$\begin{aligned} F &= \text{The cube root of unity that lies in the 2nd Quadrant of the Argand plane.} \\ A &= \text{The 5th root of unity that lies in the 4th quadrant of the Argand plane} \\ M &= \text{cis}(48^\circ) \end{aligned}$$

Find $\dfrac{FA}{M}$.

QUESTION 15

Triangle ABC has side lengths of $\tan(A), \tan(B), \tan(C)$ opposite of angles $\angle A$, $\angle B$, and $\angle C$ respectively. Let P be the area between the incircle and the circumcircle of triangle ABC.

Triangle DEF has one side of length 12 with opposite angle of $60°$. Let Q be the length of the largest latus rectum of a parabola given that the vertex and the focus of the parabola are on the circumcircle of $\triangle DEF$.

If a triangle has sides of length P and Q with an included angle of $30°$, what is its area?

QUESTION 15

Triangle ABC has side lengths of $\tan(A), \tan(B), \tan(C)$ opposite of angles $\angle A$, $\angle B$, and $\angle C$ respectively. Let P be the area between the incircle and the circumcircle of triangle ABC.

Triangle DEF has one side of length 12 with opposite angle of $60°$. Let Q be the length of the largest latus rectum of a parabola given that the vertex and the focus of the parabola are on the circumcircle of $\triangle DEF$.

If a triangle has sides of length P and Q with an included angle of $30°$, what is its area?